

Szegő via Jacobi

A. Böttcher and H. Widom*

For Bernd Silbermann on His 65th Birthday

At present there exist numerous different approaches to results on Toeplitz determinants of the type of Szegő's strong limit theorem. The intention of this paper is to show that Jacobi's theorem on the minors of the inverse matrix remains one of the most comfortable tools for tackling the matter. We repeat a known proof of the Borodin-Okounkov formula and thus of the strong Szegő limit theorem that is based on Jacobi's theorem. We then use Jacobi's theorem to derive exact and asymptotic formulas for Toeplitz determinants generated by functions with nonzero winding number. This derivation is new and completely elementary.

1 Introduction

In [9], Carey and Pincus employ heavy machinery to establish a formula for Toeplitz determinants generated by functions with nonvanishing winding number, and their paper begins with the words “Jacobi's theorem on the conjugate minors of the adjugate matrix formed from the cofactors of the Toeplitz determinant has been the main tool of previous attempts to generalize the classical strong Szegő limit theorem.” The purpose of the present paper is to demonstrate that Jacobi's theorem remains a perfect tool for deriving Szegő's theorem (which is no new message) and for generalizing the theorem to the case of nonvanishing winding number (which seems to be not widely known).

Let \mathbf{T} be the complex unit circle and let $f : \mathbf{T} \rightarrow \mathbf{C} \setminus \{0\}$ be a continuous function. We denote $\gamma \in \mathbf{Z}$ the winding number of f about the origin. So $f(t) = t^\gamma a(t)$ ($t \in \mathbf{T}$) where a has no zeros on \mathbf{T} and winding number zero. We define the Fourier coefficients f_k ($k \in \mathbf{Z}$) of f by

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ik\theta} d\theta$$

and consider the $n \times n$ Toeplitz matrices $T_n(f) := (f_{j-k})_{j,k=1}^n$ and their determinants $D_n(f) := \det T_n(f)$. We are interested in exact and asymptotic formulas for $D_n(f)$.

*The research of this author was supported by National Science Foundation grant DMS-0243982.

For the sake of definiteness, we assume that a (equivalently, f) belongs to C^β with $\beta > 1/2$, which means that a has $[\beta]$ continuous derivatives and that the $[\beta]$ th derivative satisfies a Hölder condition with the exponent $\beta - [\beta]$. To avoid well known subtleties, we suppose that $\beta \notin \mathbf{N}$.

Under the above assumptions, a has a logarithm $\log a$ in C^β , and we denote the Fourier coefficients of $\log a$ by $(\log a)_k$. We define a_- and a_+ on \mathbf{T} by

$$a_-(t) = \exp \sum_{k=1}^{\infty} (\log a)_{-k} t^{-k}, \quad a_+(t) = \exp \sum_{k=0}^{\infty} (\log a)_k t^k, \quad (1)$$

and we put $G(a) := \exp(\log a)_0$. It is well known that $a_-^{\pm 1}$ and $a_+^{\pm 1}$ belong to C^β together with a (this results from the boundedness of the Cauchy singular integral operator on C^β for $\beta \notin \mathbf{N}$). Clearly, $a = a_- a_+$. This representation is called a Wiener-Hopf factorization of a . The main actors in the following are the two functions

$$b = a_- a_+^{-1}, \quad c = a_+ a_-^{-1}. \quad (2)$$

Notice that $b \in C^\beta$, $c \in C^\beta$, and $bc = 1$.

For a continuous function φ on \mathbf{T} , we define the infinite Toeplitz matrix $T(\varphi)$ and the infinite Hankel matrix $H(\varphi)$ by $T(\varphi) := (\varphi_{j-k})_{j,k=1}^{\infty}$ and $H(\varphi) := (\varphi_{j+k-1})_{j,k=1}^{\infty}$. These two matrices induce bounded linear operators on $\ell^2(\mathbf{N})$ whose (operator) norms satisfy $\|T(\varphi)\| = \|\varphi\|_\infty$ and $\|H(\varphi)\| \leq \|\varphi\|_\infty$, where $\|\cdot\|_\infty$ is the norm in $L^\infty(\mathbf{T})$. We also define $\tilde{\varphi}$ by $\tilde{\varphi}(t) = \varphi(1/t)$ for $t \in \mathbf{T}$. Then $H(\tilde{\varphi}) = (\varphi_{-j-k+1})_{j,k=1}^{\infty}$. Let \mathcal{P}_n be the linear space of all trigonometric polynomials of degree at most n . If $\varphi \in C^\beta$, then there are $p_n \in \mathcal{P}_n$ (the polynomials of best uniform approximation) such that $\|\varphi - p_n\|_\infty = O(n^{-\beta})$. It follows that the n th singular number s_n of $H(\varphi)$ satisfies

$$s_n \leq \|H(\varphi) - H(p_n)\| = \|H(\varphi - p_n)\| \leq \|\varphi - p_n\|_\infty = O(n^{-\beta}),$$

which implies that $H(\varphi)$ is a Hilbert-Schmidt operator for $\beta > 1/2$. Consequently, the product $H(b)H(\tilde{c})$ is a trace class operator and the determinant $\det(I - H(b)H(\tilde{c}))$ is well-defined. We finally denote by P_k and Q_k the projections given by

$$\begin{aligned} P_k &: (x_1, x_2, \dots) \mapsto (x_1, \dots, x_k, 0, 0, \dots), \\ Q_k &: (x_1, x_2, \dots) \mapsto (0, \dots, 0, x_{k+1}, x_{k+2}, \dots). \end{aligned}$$

Here are the results we want to prove in this paper.

Theorem 1.1 (Borodin-Okounkov formula) *The operator $I - H(b)H(\tilde{c})$ is invertible and*

$$D_n(a) = G(a)^n \frac{\det(I - Q_n H(b) H(\tilde{c}) Q_n)}{\det(I - H(b) H(\tilde{c}))}$$

for all $n \geq 1$.

Theorem 1.2 (Szegő's strong limit theorem) *We have*

$$D_n(a) = G(a)^n E(a)(1 + O(n^{1-2\beta}))$$

where

$$\begin{aligned} E(a) &= 1/\det(I - H(b)H(\tilde{c})) = \det T(a)T(a^{-1}) \\ &= \exp \sum_{k=1}^{\infty} k(\log a)_k(\log a)_{-k} = \exp \sum_{k=1}^{\infty} k(\log b)_k(\log c)_{-k}. \end{aligned}$$

Theorem 1.3 *If $\kappa > 0$ and the matrix $T_{n+\kappa}(a)$ is invertible, then the two operators $I - H(b)H(\tilde{c})Q_{n+\kappa}$ and $I - H(b)Q_nH(\tilde{c})Q_\kappa$ are invertible and*

$$D_n(t^{-\kappa}a) = (-1)^{n\kappa} D_{n+\kappa}(a) F_{n,\kappa}(a)$$

where

$$F_{n,\kappa}(a) = \det P_\kappa T(t^{-n}) \left(I - H(b)H(\tilde{c})Q_{n+\kappa} \right)^{-1} T(b)P_\kappa \quad (3)$$

$$= \det P_\kappa \left(I - H(b)Q_nH(\tilde{c})Q_\kappa \right)^{-1} T(t^{-n}b)P_\kappa. \quad (4)$$

Theorem 1.4 (Fisher, Hartwig, Silbermann et al.) *If $\kappa > 0$, then*

$$F_{n,\kappa}(a) = \det T_\kappa(t^{-n}b) + O(n^{-3\beta}) \quad (5)$$

and thus

$$D_n(t^{-\kappa}a) = (-1)^{n\kappa} G(a)^{n+\kappa} E(a) \left(\det T_\kappa(t^{-n}b) + O(n^{-3\beta}) \right) \left(1 + O(n^{1-2\beta}) \right). \quad (6)$$

Proofs and comments on these theorems are in Sections 3, 4, 5, 6. In the following Section 2 we recall Jacobi's theorem, and Section 7 contains additional material.

2 Jacobi's theorem

Let C be an $m \times m$ matrix. For $i_1 < \dots < i_s$ and $k_1 < \dots < k_s$, we denote by $C \begin{pmatrix} i_1 & \dots & i_s \\ k_1 & \dots & k_s \end{pmatrix}$ the determinant of the submatrix of C that is formed by the intersection of the rows i_1, \dots, i_s and the columns k_1, \dots, k_s . We also define the indices $j'_1 < \dots < j'_{m-s}$ by $\{j'_1, \dots, j'_{m-s}\} := \{1, \dots, m\} \setminus \{j_1, \dots, j_s\}$.

Theorem 2.1 (Jacobi) *If A is an invertible $m \times m$ matrix, then*

$$A^{-1} \begin{pmatrix} i_1 & \dots & i_s \\ k_1 & \dots & k_s \end{pmatrix} = (-1)^{\sum_{r=1}^s (i_r + k_r)} A \begin{pmatrix} k'_1 & \dots & k'_{m-s} \\ i'_1 & \dots & i'_{m-s} \end{pmatrix} / \det A.$$

A proof is in [13], for example. The following consequence of Theorem 2.1 is from [4].

Corollary 2.2 *If K is a trace class operator and $I - K$ is invertible, then*

$$\det P_n(I - K)^{-1}P_n = \frac{\det(I - Q_n K Q_n)}{\det(I - K)}$$

for all $n \geq 1$.

Proof. If $m > n$ is sufficiently large, then $A := I_{m \times m} - P_m K P_m$ is invertible together with $I - K$. Theorem 2.1 with $\{i_1, \dots, i_n\} = \{k_1, \dots, k_n\} = \{1, \dots, n\}$ applied to the $m \times m$ matrix A yields

$$\det P_n(I_{m \times m} - P_m K P_m)^{-1}P_n = \frac{\det(I_{(m-n) \times (m-n)} - Q_n P_m K P_m Q_n)}{\det(I_{m \times m} - P_m K P_m)},$$

which is equivalent to

$$\det P_n(I - P_m K P_m)^{-1}P_n = \frac{\det(I - Q_n P_m K P_m Q_n)}{\det(I - P_m K P_m)}. \quad (7)$$

Since $P_m K P_m \rightarrow K$ in the trace norm as $m \rightarrow \infty$ and the determinant is continuous on identity minus trace class ideal, we may in (7) pass to the limit $m \rightarrow \infty$ to get the desired formula.

Here is another corollary of Jacobi's theorem. It was Fisher and Hartwig [11], [12] who were the first to write down this corollary and to recognize that it is the key to treating the case of nonvanishing winding number.

Corollary 2.3 *Let $\kappa > 0$ and suppose $T_{n+\kappa}(a)$ is invertible. Then*

$$D_n(t^{-\kappa}a) = (-1)^{n\kappa} D_{n+\kappa}(a) \det(P_{n+\kappa} - P_n) T_{n+\kappa}^{-1}(a) P_\kappa.$$

Proof. This is immediate from Theorem 2.1 with $A = T_{n+\kappa}(a)$ and

$$\begin{aligned} \det(P_{n+\kappa} - P_n) T_{n+\kappa}^{-1}(a) P_\kappa &= A^{-1} \begin{pmatrix} n+1 & \dots & n+\kappa \\ 1 & \dots & \kappa \end{pmatrix}, \\ D_n(t^{-\kappa}a) &= A \begin{pmatrix} \kappa+1 & \dots & \kappa+n \\ 1 & \dots & n \end{pmatrix}, \\ (-1)^{(n+\kappa+1)\kappa} &= (-1)^{n\kappa}. \end{aligned}$$

3 The Borodin-Okounkov formula

Theorem 1.1 was established by Borodin and Okounkov in [2]. Later it turned out that (for positive functions a) it was already in Geronimo and Case's paper [14]. The original proofs in [2], [14] are quite complicated. Simpler proofs were subsequently found in [1], [3], [4]. See also [9]. Here is the proof from [4], which is based on Jacobi's theorem.

We apply Corollary 2.2 to the trace class operator $K = H(b)H(\tilde{c})$. The operator

$$\begin{aligned} I - K &= T(bc) - H(b)H(\tilde{c}) = T(b)T(c) \\ &= T(a_-)T(a_+^{-1})T(a_-^{-1})T(a_+) = T(a_-)T^{-1}(a_-a_+)T(a_+) \end{aligned}$$

has the inverse

$$(I - K)^{-1} = T(a_+^{-1})T(a_-a_+)T(a_-^{-1})$$

and hence Corollary 2.2 yields

$$\det P_n T(a_+^{-1})T(a_-a_+)T(a_-^{-1})P_n = \frac{\det(I - Q_n H(b)H(\tilde{c})Q_n)}{\det(I - H(b)H(\tilde{c}))}. \quad (8)$$

Taking into account that

$$P_n T(a_+^{-1})T(a_-a_+)T(a_-^{-1})P_n = P_n T(a_+^{-1})P_n T(a_-a_+)P_n T(a_-^{-1})P_n$$

and that $P_n T(a_+^{-1})P_n$ and $P_n T(a_-^{-1})P_n$ are triangular with $1/G(a)$ and 1, respectively, on the main diagonal, we see that the left-hand side of (8) equals $D_n(a)/G(a)^n$.

4 The strong Szegő limit theorem

We now prove Theorem 1.2. As $H(b)H(\tilde{c})$ is in the trace class and $Q_n = Q_n^*$ goes strongly to zero, we have $\det(I - Q_n H(b)H(\tilde{c})Q_n) = 1 + o(1)$. Thus, Theorem 1.1 immediately gives

$$D_n(a) = G(a)^n E(a)(1 + o(1)) \quad \text{with} \quad E(a) = 1/\det(I - H(b)H(\tilde{c})).$$

To make the $o(1)$ precise, we proceed as in [5], [6]. The ℓ th singular number s_ℓ of $Q_n H(b)$ can be estimated by

$$s_\ell \leq \|Q_n H(b) - Q_n H(p_{n+\ell})\| = \|Q_n H(b - p_{n+\ell})\| \leq \|b - p_{n+\ell}\|_\infty$$

where $p_{n+\ell}$ is any polynomial in $\mathcal{P}_{n+\ell}$. There are such polynomials with $\|b - p_{n+\ell}\|_\infty = O((n + \ell)^{-\beta})$. This shows that the squared Hilbert-Schmidt norm of $Q_n H(b)$ is

$$\sum_{\ell=0}^{\infty} s_\ell^2 = O\left(\sum_{\ell=0}^{\infty} (n + \ell)^{-2\beta}\right) = O(n^{1-2\beta}).$$

Thus, the Hilbert-Schmidt norm of $Q_n H(b)$ is $O(n^{1/2-\beta})$. The same is true for the operator $H(\tilde{c})Q_n$. Consequently, the trace norm of $Q_n H(b)H(\tilde{c})Q_n$ is $O(n^{1-2\beta})$, which implies that

$$D_n(a) = G(a)^n E(a)(1 + O(n^{1-2\beta})) \quad \text{with} \quad E(a) = 1/\det(I - H(b)H(\tilde{c})).$$

We are left with the alternative expressions for $E(a)$. We start with

$$\begin{aligned} 1/\det(I - H(b)H(\tilde{c})) &= 1/\det T(b)T(c) = \det T^{-1}(c)T^{-1}(b) \\ &= \det T(a_+^{-1})T(a_-)T(a_+)T(a_-^{-1}) = \det T(a_-)T(a_+)T(a_-^{-1})T(a_+^{-1}). \end{aligned} \quad (9)$$

This equals

$$\det T(a_- a_+)T(a_-^{-1} a_+^{-1}) = \det T(a)T(a^{-1}).$$

On the other hand, (9) is

$$\det e^{T(\log a_-)} e^{T(\log a_+)} e^{-T(\log a_-)} e^{-T(\log a_+)} \quad (10)$$

and the Pincus-Helton-Howe formula [15], [16] (an easy proof of which was recently found by Ehrhardt [10]) says that

$$\det e^A e^B e^{-A} e^{-B} = e^{\text{tr}(AB-BA)}$$

whenever A and B are bounded and $AB-BA$ is in the trace class. Thus, (10) becomes

$$\begin{aligned} &\exp \text{tr} \left(T(\log a_-)T(\log a_+) - T(\log a_+)T(\log a_-) \right) \\ &= \exp \text{tr} H(\log a_+)H((\log a_-)^{\sim}) = \exp \sum_{k=1}^{\infty} k(\log a_+)_k(\log a_-)_{-k} \\ &= \exp \sum_{k=1}^{\infty} k(\log a)_k(\log a)_{-k} = \exp \sum_{k=1}^{\infty} k(\log b)_k(\log c)_{-k}. \end{aligned}$$

Theorem 1.2 is completely proved.

The treatment of the constant $E(a)$ given here is from [19]. For reviews of the gigantic development from Szegő's original version of his strong limit theorem [18] up to the present we refer to the books [6] and [17].

5 The exact formula for nonzero winding numbers

To prove Theorem 1.3 we use Corollary 2.3 of Jacobi's theorem. Thus, we must show that

$$\det(P_{n+\kappa} - P_n)T_{n+\kappa}^{-1}(a)P_{\kappa} =: F_{n,\kappa}(a)$$

is given by (3) and (4).

We put $K := H(b)H(\tilde{c})$, $m := n + \kappa$, $\Delta_n^\kappa := P_{n+\kappa} - P_n$. In [3] it was shown (in an elementary way) that the invertibility of $T_m(a)$ implies that $I - Q_m K Q_m$ is invertible and that

$$T_m^{-1}(a) = P_m T(a_+^{-1}) \left(I - T(c) Q_m (I - Q_m K Q_m)^{-1} Q_m T(b) \right) T(a_-^{-1}) P_m$$

(to get conformity with [3] note that $P_m T(a_+^{-1}) P_m = P_m T(a_+^{-1})$ and $P_m T(a_-^{-1}) P_m = T(a_-^{-1}) P_m$). We multiply this identity from the right by P_κ and from the left by Δ_n^κ . Since

$$T(a_-^{-1}) P_m P_\kappa = T(a_-^{-1}) P_\kappa = P_\kappa T(a_-^{-1}) P_\kappa$$

and

$$\begin{aligned} \Delta_n^\kappa P_m T(a_+^{-1}) &= \Delta_n^\kappa T(a_+^{-1}) = \Delta_n^\kappa T(a_-^{-1}) T(a_-) T(a_+^{-1}) \\ &= \Delta_n^\kappa T(a_-^{-1}) T(b) = \Delta_n^\kappa T(a_-^{-1}) \Delta_n^\kappa T(b), \end{aligned}$$

we arrive at the formula

$$\begin{aligned} \det \Delta_n^\kappa T_m^{-1}(a) P_\kappa &= \det \Delta_n^\kappa T(a_-^{-1}) \Delta_n^\kappa \cdot \det P_\kappa T(a_-^{-1}) P_\kappa \\ &\quad \times \det \Delta_n^\kappa T(b) \left(I - T(c) Q_m (I - Q_m K Q_m)^{-1} Q_m T(b) \right) P_\kappa. \end{aligned}$$

As the matrix $T(a_-^{-1})$ is triangular with 1 on the main diagonal, we have

$$\det \Delta_n^\kappa T(a_-^{-1}) \Delta_n^\kappa = \det P_\kappa T(a_-^{-1}) P_\kappa = 1 \quad (11)$$

and are therefore left with the determinant of

$$\Delta_n^\kappa T(b) \left(I - T(c) Q_m (I - Q_m K Q_m)^{-1} Q_m T(b) \right) P_\kappa. \quad (12)$$

Taking into account that $T(b)T(c) = T(bc) - H(b)H(\tilde{c}) = I - K$ and $\Delta_n^\kappa Q_m = 0$, we obtain that (12) equals

$$\begin{aligned} &\Delta_n^\kappa T(b) P_\kappa - \Delta_n^\kappa (I - K) Q_m (I - Q_m K Q_m)^{-1} Q_m T(b) P_\kappa \\ &= \Delta_n^\kappa T(b) P_\kappa + \Delta_n^\kappa K Q_m (I - Q_m K Q_m)^{-1} Q_m T(b) P_\kappa \\ &= \Delta_n^\kappa \left(I + K Q_m (I - Q_m K Q_m)^{-1} Q_m \right) T(b) P_\kappa. \end{aligned} \quad (13)$$

Since

$$(I - Q_m K Q_m)^{-1} Q_m (I - Q_m K Q_m) = (I - Q_m K Q_m)^{-1} (I - Q_m K Q_m) Q_m = Q_m,$$

we get $(I - Q_m K Q_m)^{-1} Q_m = Q_m (I - Q_m K Q_m)^{-1}$ and thus

$$I + K Q_m (I - Q_m K Q_m)^{-1} Q_m = I + K Q_m (I - Q_m K Q_m)^{-1}. \quad (14)$$

We have $I - KQ_m = (I - Q_m KQ_m)(I - P_m KQ_m)$ and the operators $I - Q_m KQ_m$ and $I - P_m KQ_m$ are invertible; note that $(I - P_m KQ_m)^{-1} = I + P_m KQ_m$. Consequently, $I - KQ_m$ is also invertible. It follows that (14) is

$$\begin{aligned} & (I - Q_m KQ_m)(I - Q_m KQ_m)^{-1} + KQ_m(I - Q_m KQ_m)^{-1} \\ &= (I - Q_m KQ_m + KQ_m)(I - Q_m KQ_m)^{-1} \\ &= (I + P_m KQ_m)(I - Q_m KQ_m)^{-1} \\ &= (I - P_m KQ_m)^{-1}(I - Q_m KQ_m)^{-1} = (I - KQ_m)^{-1}. \end{aligned} \quad (15)$$

In summary, (13) is $\Delta_n^\kappa(I - KQ_m)^{-1}T(b)P_\kappa$ and we have proved the theorem with

$$F(n, \kappa) = \det \Delta_n^\kappa(I - KQ_{n+\kappa})^{-1}T(b)P_\kappa.$$

The operator $T(t^{-k})$ sends (x_1, x_2, \dots) to $(x_{k+1}, x_{k+2}, \dots)$. It follows that $\Delta_n^\kappa = P_\kappa T(t^{-n})$, which yields (3).

The matrix $I - KQ_{n+\kappa}$ is of the form

$$\begin{pmatrix} I_{(n+\kappa) \times (n+\kappa)} & * \\ 0 & B \end{pmatrix},$$

and since $I - KQ_{n+\kappa}$ is invertible, the matrix B must also be invertible. The matrix

$$M := T(t^{-n})(I - KQ_{n+\kappa})T(t^n) \quad (16)$$

results from $I - KQ_{n+\kappa}$ by deleting the first n rows and first n columns. Consequently, M has the form

$$\begin{pmatrix} I_{\kappa \times \kappa} & * \\ 0 & B \end{pmatrix},$$

and the invertibility of B implies that M is invertible. Since $T(t^k)T(t^{-k}) = Q_k$ and hence

$$MT(t^{-n}) = T(t^{-n})(I - KQ_{n+\kappa})Q_n = T(t^{-n})(I - KQ_{n+\kappa}),$$

we get $T(t^{-n})(I - KQ_{n+\kappa})^{-1} = M^{-1}T(t^{-n})$. Inserting this in (3) we arrive at the formula

$$F(n, \kappa) = \det P_\kappa M^{-1}T(t^{-n})T(b)P_\kappa = \det P_\kappa M^{-1}T(t^{-n}b)P_\kappa.$$

Finally, the identity $T(t^{-k})H(\varphi) = H(\varphi)T(t^k)$ shows that

$$\begin{aligned} M &= I - T(t^{-n})H(b)H(\tilde{c})T(t^{n+\kappa})T(t^{-n-\kappa})T(t^n) \\ &= I - T(t^{-n})H(b)H(\tilde{c})T(t^n)T(t^\kappa)T(t^{-\kappa}) \\ &= I - H(b)T(t^n)T(t^{-n})H(\tilde{c})T(t^\kappa)T(t^{-\kappa}) \\ &= I - H(b)Q_n H(\tilde{c})Q_\kappa, \end{aligned}$$

which gives (4) and completes the proof of Theorem 1.3.

A result like Theorem 1.3 appeared probably first in [20]. Given a set $E \subset \mathbf{Z}$, we denote by P_E the projection on $L2(\mathbf{T})$ defined by

$$P_E : \sum_{k \in \mathbf{Z}} x_k t^k \mapsto \sum_{k \in E} x_k t^k,$$

and for a function φ on \mathbf{T} , we denote the operator of multiplication by φ on $L2(\mathbf{T})$ also by φ . Let U and V be the operators on $L2(\mathbf{T})$ given by

$$U := P_{\{1,2,\dots\}} t^{-n-\kappa+1} b, \quad V := P_{\{-1,-2,\dots\}} t^{n+\kappa-1} c.$$

One can show that $I - VU$ is invertible. Put $Y = (Y_{ij})_{i,j=0}^{\kappa-1}$ with

$$Y_{ij} = P_{\{-i\}} t^{-n-\kappa+1} b (I - VU)^{-1} P_{\{j\}}.$$

Lemma 3.2 of [20] says that

$$\det \Delta_n^{\kappa} T_{n+\kappa}^{-1}(a) P_{\kappa} = (-1)^{\kappa} \det Y, \quad (17)$$

which together with Corollary 2.3 yields

$$D_n(t^{-\kappa} a) = (-1)^{n\kappa} D_{n+\kappa}(a) (-1)^{\kappa} \det Y.$$

Clearly, this highly resembles Theorem 1.3. In Remark 7.2 we will show that the right-hand side of (17) indeed coincides with (4).

Carey and Pincus [9] state that

$$D_n(t^{-\kappa} a) = (-1)^{n\kappa} G(a)^{n+\kappa} E(a) \tilde{F}_{n,\kappa}(a) (1 + O(n^{1-2\beta})) \quad (18)$$

with

$$\tilde{F}_{n,\kappa}(a) = \det P_{\kappa} \left(I - H(b) Q_{n-\kappa} H(\tilde{c}) \right)^{-1} T(t^{-n} b) P_{\kappa}. \quad (19)$$

The proof of (18), (19) given in [9] is complicated and based on the methods developed in these authors' work [7], [8], [9]. We will return to (18), (19) in Remark 7.3.

6 The asymptotic formula for nonzero winding numbers

Theorem 1.4 is an easy consequence of Theorem 1.3. Since $\|H(b) Q_n H(\tilde{c}) Q_{\kappa}\| \rightarrow 0$ as $n \rightarrow \infty$, we obtain that

$$F_{n,\kappa}(a) = \det \left[P_{\kappa} T(t^{-n} b) P_{\kappa} + \sum_{k=1}^{\infty} P_{\kappa} \left(H(b) Q_n H(\tilde{c}) Q_{\kappa} \right)^k T(t^{-n} b) P_{\kappa} \right] \quad (20)$$

for all sufficiently large n . We know that there are polynomials p_n and q_n in $\mathcal{P}_{n-\kappa-1}$ such that $\|b - p_n\|_{\infty} = O(n^{-\beta})$ and $\|c - q_n\|_{\infty} = O(n^{-\beta})$. It follows that

$$\begin{aligned} \|H(b) Q_n\| &= \|H(b - p_n) Q_n\| \leq \|b - p_n\|_{\infty} = O(n^{-\beta}), \\ \|Q_n H(\tilde{c})\| &= \|Q_n H(\tilde{c} - q_n)\| \leq \|c - q_n\|_{\infty} = O(n^{-\beta}), \\ \|T(t^{-n} b) P_{\kappa}\| &= \|T(t^{-n} (b - p_n)) P_{\kappa}\| = \|b - p_n\|_{\infty} = O(n^{-\beta}). \end{aligned}$$

Since P_κ is a trace class operator, the sum in (20) is $O(n^{-3\beta})$ in the trace norm, which implies the claim of Theorem 1.4.

We remark that a result close to Theorem 1.4 was already established by Fisher and Hartwig [11], [12] using different methods. Theorem 1.4 as it is stated, a formula similar to (20), and the estimates via $\|b - p_n\|_\infty$ and $\|c - q_n\|_\infty$ used above are due to Silbermann and one of the authors [5].

7 Remarks

Here are a few additional issues.

Remark 7.1 The proof of Theorem 1.3 given in Section 5 was done under the minimal assumption that $T_{n+\kappa}(a)$ be invertible. The proof can be simplified if one is satisfied by the formula for sufficiently large n only. Indeed, the operator $K = H(b)H(\tilde{c})$ is compact and hence $\|KQ_m\| \rightarrow 0$ as $m \rightarrow \infty$. It follows that $\|KQ_m\| < 1$ whenever $m = n + \kappa$ is large enough, and for these m we can replace all between (13) and (15) by the simple series argument

$$\begin{aligned} & I + KQ_m(I - Q_mKQ_m)^{-1}Q_m \\ &= I + KQ_m + KQ_mKQ_m + KQ_mKQ_mKQ_m + \dots \\ &= (I - KQ_m)^{-1}. \end{aligned}$$

Moreover, if $\|KQ_m\| < 1$ then the invertibility of the operator (16) is obvious and we can omit the piece of the proof dedicated to the invertibility of (16).

Remark 7.2 We prove that the right-hand side of (17) is the same as (4). We identify $L2(\mathbf{T})$ with $\ell^2(\mathbf{Z})$ in the natural fashion and think of operators on $L2(\mathbf{T})$ as acting by infinite matrices on $\ell^2(\mathbf{Z})$. Let $m := n + \kappa$. The matrices of U and V have the entries

$$U_{ij} = \begin{cases} b_{i-j+m-1} & \text{if } i > 0, \\ 0 & \text{if } i \leq 0, \end{cases} \quad V_{ij} = \begin{cases} c_{i-j-m+1} & \text{if } i < 0, \\ 0 & \text{if } i \geq 0, \end{cases}$$

and the matrix of the multiplication operator $B := t^{-m+1}b$ has i, j entry $b_{i-j+m-1}$. The i, j entry of the product VU equals

$$\sum_{k>0} \tilde{c}_{-i+k+m-1} b_{k-j+m-1}$$

for $i < 0$ and is 0 for $i \geq 0$. If we set

$$H_{ij} = (VU)_{ij} \quad (j < 0), \quad L_{ij} = (VU)_{ij} \quad (j \geq 0),$$

with both equal to 0 when $i \geq 0$, then the operator $I - VU$ has the matrix representation

$$\begin{pmatrix} I - H & -L \\ 0 & I \end{pmatrix} \tag{21}$$

corresponding to the decomposition $\ell^2(\mathbf{Z}) = \ell^2(\mathbf{Z}_-) \oplus \ell^2(\mathbf{Z}_+)$ with $\mathbf{Z}_- = \{-1, -2, \dots\}$ and $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$. The inverse of (21) is

$$\begin{pmatrix} (I - H)^{-1} & (I - H)^{-1}L \\ 0 & I \end{pmatrix}.$$

Hence the i, j entry of Y is

$$Y_{ij} = (BP_-(I - H)^{-1}L)_{-i,j} + b_{-i-j+m-1},$$

where $P_{\pm} := P_{\mathbf{Z}_{\pm}}$. Here i and j run from 0 to $\kappa - 1$. Now replace i by $\kappa - i - 1$. The new index also runs from 0 to $\kappa - 1$. Thus,

$$Y_{\kappa-i-1,j} = (BP_-(I - H)^{-1}L)_{-\kappa+i+1,j} + b_{-\kappa+i-j+m}.$$

Let J be given on $\ell^2(\mathbf{Z})$ by $(Jx)_k = x_{k-1}$. Then $J^2 = I$ and $P_-J = JP_+$. Consequently,

$$\begin{aligned} Y_{\kappa-i-1,j} &= (BP_-J(I - JHJ)^{-1}JL)_{-\kappa+i+1,j} + b_{-\kappa+i-j+m} \\ &= (BJP_+(I - JHJ)^{-1}JL)_{-\kappa+i+1,j} + b_{-\kappa+i-j+m}. \end{aligned}$$

The matrix $(b_{-\kappa+i-j+m})$ at the end delivers $T(t^{\kappa-m}b)$. Next, BJ has $-\kappa + i + 1, j$ entry $b_{i+j+m-\kappa+1}$, which is the i, j entry of $H(t^{\kappa-m}b)$. The i, j entry of JHJ is

$$\sum_{k \geq 0} \tilde{c}_{i+k+m+1} b_{k+j+m+1} = \sum_{k \geq \kappa} \tilde{c}_{i+k+m-\kappa+1} b_{k+j+m-\kappa+1}$$

and so the operator itself is $H(t^{\kappa-m}\tilde{c})Q_{\kappa}H(t^{\kappa-m}b)$. Finally, the i, j entry of JL is equal to

$$\sum_{k \geq 0} \tilde{c}_{i+k+m+1} b_{k-j+m} = \sum_{k \geq \kappa} \tilde{c}_{i+k-\kappa+m+1} b_{k-\kappa-j+m}$$

whence $JL = H(t^{\kappa-m}\tilde{c})Q_{\kappa}T(t^{\kappa-m}b)$. Let $D := H(t^{\kappa-m}b)$ and $C := H(t^{\kappa-m}\tilde{c})$. We have shown that $Y_{\kappa-i-1,j}$ is the i, j entry of

$$\begin{aligned} &\left(D(I - CQ_{\kappa}D)^{-1}CQ_{\kappa} + I \right) T(t^{\kappa-m}b) \\ &= (I - DCQ_{\kappa})^{-1} T(t^{\kappa-m}b) \\ &= \left(I - H(t^{\kappa-m}b)H(t^{\kappa-m}\tilde{c})Q_{\kappa} \right)^{-1} T(t^{\kappa-m}b) \\ &= \left(I - H(b)Q_{m-\kappa}H(\tilde{c})Q_{\kappa} \right)^{-1} T(t^{\kappa-m}b) \\ &= \left(I - H(b)Q_nH(\tilde{c})Q_{\kappa} \right)^{-1} T(t^{-n}b). \end{aligned}$$

It follows that $(-1)^{\kappa} \det(Y_{ij}) = \det(Y_{\kappa-i-1,j})$ equals

$$\det P_{\kappa} \left(I - H(b)Q_nH(\tilde{c})Q_{\kappa} \right)^{-1} T(t^{-n}b) P_{\kappa},$$

as desired.

Remark 7.3 We show that (18), (19) are consistent with Theorem 1.3. Let first

$$M_n = I - H(b)Q_n H(\tilde{c})Q_\kappa, \quad R_n = I - H(b)Q_n H(\tilde{c}), \quad T_n = T(t^{-n}b).$$

We have $M_n = R_n + H(b)Q_n H(\tilde{c})P_\kappa =: R_n + Z_n$. Using best approximation of b and c as above, we get $\|Z_n\| = O(n^{-2\beta})$, and it is clear that $\|R_n^{-1}\| = O(1)$. The identity $M_n^{-1} = (I + R_n^{-1}Z_n)^{-1}R_n^{-1}$ implies that

$$P_\kappa M_n^{-1}T_n P_\kappa = P_\kappa(I + R_n^{-1}Z_n)^{-1}P_\kappa R_n^{-1}T_n P_\kappa + P_\kappa(I + R_n^{-1}Z_n)^{-1}Q_\kappa R_n^{-1}T_n P_\kappa,$$

and the second term on the right is zero because $P_\kappa(I + R_n^{-1}Z_n)^{-1}$ has P_κ at the end. It follows that

$$\det P_\kappa M_n^{-1}T_n P_\kappa = \det P_\kappa R_n^{-1}T_n P_\kappa (1 + O(n^{-2\beta})),$$

or equivalently,

$$F_{n,\kappa}(a) = \det P_\kappa \left(I - H(b)Q_n H(\tilde{c}) \right)^{-1} T(t^{-n}b) P_\kappa (1 + O(n^{-2\beta})). \quad (22)$$

To change the Q_n to $Q_{n-\kappa}$, let

$$S_n = I - H(b)Q_{n-\kappa} H(\tilde{c}), \quad X_n = H(b)(Q_{n-\kappa} - Q_n)H(\tilde{c}).$$

Then $R_n = S_n + X_n$, $\|X_n\| = O(n^{-2\beta})$, $\|S_n^{-1}\| = O(1)$, and

$$P_\kappa R_n^{-1}T_n P_\kappa = P_\kappa(I + S_n^{-1}X_n)^{-1}P_\kappa S_n^{-1}T_n P_\kappa + P_\kappa(I + S_n^{-1}X_n)^{-1}Q_\kappa S_n^{-1}T_n P_\kappa.$$

This time the second term on the right does not disappear and hence all we can say is that

$$\det P_\kappa R_n^{-1}T_n P_\kappa = \det P_\kappa S_n^{-1}T_n P_\kappa + O(n^{-2\beta}).$$

Combining this and (22) we arrive at the formula $F_{n,\kappa}(a) = \tilde{F}_{n,\kappa}(a) + O(n^{-2\beta})$ and thus at

$$D_n(t^{-\kappa}a) = (-1)^{n\kappa} G(a)^{n+\kappa} E(a) (\tilde{F}_{n,\kappa}(a) + O(n^{-2\beta})) (1 + O(n^{1-2\beta})),$$

which is not yet (18) but reveals that (18) is consistent with Theorem 1.3. We emphasize that (18) is an asymptotic result while Theorem 1.3 provides us with an exact formula. Moreover, $F_{n,\kappa}(a)$ is a little better than $\tilde{F}_{n,\kappa}(a)$ since Q_n and Q_κ are “smaller” than $Q_{n-\kappa}$ and I .

Remark 7.4 We worked with the Wiener-Hopf factorization $a = a_- a_+$ specified by (1). One can do everything if one starts with an arbitrary Wiener-Hopf factorization $a = a_- a_+$. The different factorizations are all of the form $a = (\mu^{-1}a_-)(\mu a_+)$ where μ is a nonzero complex number. The functions b and c are then defined by

$$b = (\mu^{-1}a_-)(\mu a_+)^{-1} = \mu^{-2}a_- a_+^{-1}, \quad c = (\mu^{-1}a_-)^{-1}(\mu a_+) = \mu 2a_-^{-1}a_+.$$

Theorems 1.1 and 1.2 are invariant under this change. The only difference in Theorems 1.3 and 1.4 is that if we replace a_- by $\mu^{-1}a_-$ in (11), then the determinants are μ^κ and their product becomes $\mu^{2\kappa}$. Since $G(a) = G(a_-a_+)$ and $G(c) = \mu^2G(a_-a_+)$, we obtain that $\mu^2 = G(a)^{-\kappa}G(c)^\kappa$ and hence

$$\det \Delta_n^\kappa T_{n-\kappa}^{-1}(a)P_\kappa = G(a)^{-\kappa}G(c)^\kappa F_{n,\kappa}(a).$$

The invariant versions of Theorems 1.3 and 1.4 are

$$\begin{aligned} D_n(t^{-\kappa}a) &= (-1)^{n\kappa} D_{n+\kappa}(a)G(a)^{-\kappa}G(c)^\kappa F_{n,\kappa}(a) \\ &= (-1)^{n\kappa} G(a)^n E(a)G(c)^\kappa \left(\det T_\kappa(t^{-n}b) + O(n^{-3\beta}) \right) (1 + O(n^{1-2\beta})), \end{aligned}$$

where $F_{n,\kappa}(a)$ is given by (3) and (4).

Remark 7.5 Theorems 1.1 to 1.4 can be extended to block Toeplitz operators generated by $\mathbf{C}^{N \times N}$ -valued C^β -functions. In that case one has to start with two Wiener-Hopf factorizations $a = u_- u_+ = v_+ v_-$ and to put $b = v_- u_+^{-1}$, $c = u_-^{-1} v_+$. Theorem 1.1 and its proof remain in force literally. Theorem 1.2 and its proof yield the operator determinants for $E(a)$ but not the expressions in terms of the Fourier coefficients of $\log a$, $\log b$, $\log c$. In Theorems 1.3 and 1.4 one has to require that all partial indices be equal to one another. The result reads

$$\begin{aligned} & D_n \left[\begin{pmatrix} t^{-\kappa} & & \\ & \ddots & \\ & & t^{-\kappa} \end{pmatrix} a \right] \\ &= (-1)^{n\kappa N} D_{n+\kappa}(a)G(a)^{-\kappa}G(c)^\kappa \det P_\kappa \left(I - H(b)Q_n H(\tilde{c})Q_\kappa \right)^{-1} T(t^{-n}b)P_\kappa \\ &= (-1)^{n\kappa N} G(a)^n E(a)G(c)^\kappa \left(\det T_\kappa(t^{-n}b) + O(n^{-3\beta}) \right) (1 + O(n^{1-2\beta})). \end{aligned}$$

For details see [3] and [6].

Remark 7.6 The case of positive winding numbers can be reduced to negative winding numbers by passage to transposed matrices because $D_n(t^\kappa a) = D_n(t^{-\kappa} \tilde{a})$. Let $a = a_- a_+$ be any Wiener-Hopf factorization. We denote the functions associated with \tilde{a} through (2) by b_* and c_* :

$$b_* = \widetilde{a_+} \widetilde{a_-}^{-1} = \tilde{c}, \quad c_* = \widetilde{a_-} \widetilde{a_+}^{-1} = \tilde{b}.$$

From Remark 7.4 we infer that

$$D_n(t^\kappa a) = (-1)^{n\kappa} D_{n+\kappa}(a)G(a)^{-\kappa}G(c_*)^\kappa F_{n,\kappa}(\tilde{a})$$

with $G(c_*) = G(\tilde{b}) = G(b)$ and

$$\begin{aligned} F_{n,\kappa}(\tilde{a}) &= \det P_\kappa \left(I - H(b_*)Q_n H(\tilde{c}_*)Q_\kappa \right)^{-1} T(t^{-n}b_*)P_\kappa \\ &= \det P_\kappa \left(I - H(\tilde{c})Q_n H(b)Q_\kappa \right)^{-1} T(t^{-n}\tilde{c})P_\kappa \\ &= \det P_\kappa T(t^n c) \left(I - Q_\kappa H(b)Q_n H(\tilde{c}) \right)^{-1} P_\kappa. \end{aligned}$$

References

- [1] E. L. Basor and H. Widom: On a Toeplitz determinant identity of Borodin and Okounkov. *Integral Equations Operator Theory* **37** (2000), 397–401.
- [2] A. Borodin and A. Okounkov: A Fredholm determinant formula for Toeplitz determinants. *Integral Equations Operator Theory* **37** (2000), 386–396.
- [3] A. Böttcher: One more proof of the Borodin-Okounkov formula for Toeplitz determinants. *Integral Equations Operator Theory* **41** (2001), 123–125.
- [4] A. Böttcher: On the determinant formulas by Borodin, Okounkov, Baik, Deift, and Rains. In *Toeplitz Matrices and Singular Integral Equations: Bernd Silbermann Anniversary Volume*, pp. 91–99, Oper. Theory Adv. Appl., Vol. 135, Birkhäuser, Basel 2002.
- [5] A. Böttcher and B. Silbermann: Notes on the asymptotic behavior of block Toeplitz matrices and determinants. *Math. Nachr.* **98** (1980), 183–210.
- [6] A. Böttcher and B. Silbermann: *Analysis of Toeplitz Operators*. 2nd edition, Springer, Berlin, Heidelberg, New York 2006.
- [7] R. W. Carey and J. D. Pincus: Perturbation vectors. *Integral Equations Operator Theory* **35** (1999), 271–365.
- [8] R. W. Carey and J. D. Pincus: Toeplitz operators with rational symbols, reciprocity. *Integral Equations Operator Theory* **40** (2001), 127–184.
- [9] R. W. Carey and J. D. Pincus: Steinberg symbols modulo the trace class, holonomy, and limit theorems for Toeplitz determinants. *Trans. Amer. Math. Soc.* **358** (2006), 509–551.
- [10] T. Ehrhardt: A generalization of Pincus’ formula and Toeplitz operator determinants. *Arch. Math. (Basel)* **80** (2003), 302–309.
- [11] M. E. Fisher and R. E. Hartwig: Toeplitz determinants: some applications, theorems, and conjectures. *Adv. Chem. Phys.* **15** (1968), 333–353.
- [12] M. E. Fisher and R. E. Hartwig: Asymptotic behavior of Toeplitz matrices and determinants. *Arch. Rational Mech. Anal.* **32** (1969), 190–225.
- [13] F. R. Gantmacher: *The Theory of Matrices*. Vols. 1 and 2, Chelsea, New York 1959.
- [14] J. S. Geronimo and K. M. Case: Scattering theory and polynomials orthogonal on the unit circle. *J. Math. Phys.* **20** (1979), 299–310.

- [15] J. W. Helton and R. E. Howe: Integral operators: commutators, traces, index and homology. In *Proceedings of a Conference on Operator Theory (Dalhousie Univ., Halifax, N.S., 1973)*, pp. 141–209, Lecture Notes in Math., Vol. 345, Springer, Berlin 1973.
- [16] J. D. Pincus: On the trace of commutators in the algebra of operators generated by an operator with trace class self-commutator. Unpublished manuscript, 1972.
- [17] B. Simon: *Orthogonal Polynomials on the Unit Circle. Part 1. Classical Theory*. Amer. Math. Soc. Colloquium Publications, Vol. 54, Amer. Math. Soc., Providence, RI 2005.
- [18] G. Szegő: On certain Hermitian forms associated with the Fourier series of a positive function. In *Festschrift Marcel Riesz*, pp. 222–238, Lund 1952.
- [19] H. Widom: Asymptotic behavior of block Toeplitz matrices and determinants, II. *Advances in Math.* **21** (1976), 1–29.
- [20] H. Widom: Eigenvalue distribution of nonselfadjoint Toeplitz matrices and the asymptotics of Toeplitz determinants in the case of nonvanishing index. In *Topics in Operator Theory: Ernst D. Hellinger Memorial Volume*, pp. 387–421, Oper. Theory Adv. Appl., Vol. 48, Birkhäuser, Basel 1990.

A. Böttcher
 Fakultät für Mathematik
 Technische Universität Chemnitz
 09107 Chemnitz
 Germany
 aboettch@mathematik.tu-chemnitz.de

H. Widom
 Department of Mathematics
 University of California
 Santa Cruz, CA 95064
 USA
 widom@math.ucsc.edu

MSC 2000: 47B35